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Noncoincidence of Geodesic Lengths and Hearing Elliptic Quantum Billiards

Edoh Y. Amiran[1]

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Assume that the planar region Ω has a C^1 boundary $\partial\Omega$ and is strictly convex in the sense that the tangent angle determines a point on the boundary. The lengths of invariant circles for the billiard ball map (or caustics) accumulate on $|\partial\Omega|$. It follows from direct calculations and from relations between the lengths of invariant circles and the lengths of trajectories of the billiard ball map that under mild assumptions on the lengths of some geodesics the region satisfies the strong noncoincidence condition. This condition plays a role in recovering the lengths of closed geodesics from the spectrum of the Laplacian. Asymptotics for the lengths of invariant circles and an application to ellipses are discussed. In addition; some examples regarding strong non coincidence are given.

KEY WORDS: Billiards; spectral invariants; noncoincidence; ellipse.

OVERVIEW

Sections 1–3 provide the background for the new results and examples regarding noncoincidence in Sections 4–6 and Corollary 7. Sections 7 and 8 illustrate the use of noncoincidence with invariants.

1. THE BILLIARD BALL MAP

Let Ω denote a bounded domain in the plane. A geodesic in such a domain is a continuous curve consisting of line segments with endpoints on the boundary. $\partial\Omega$, so that two segments meeting at a point p form equal angles with the tangent to $\partial\Omega$ at p.

The collection of unit vectors over the boundary pointing into the domain is denoted by $S_{\partial\Omega}\Omega$ (It is homeomorphic to a cylinder.) The

[1] Mathematics, Western Washington University, Bellingham, Washington 98225-9063; e-mail: edoh@wwu.edu.

billiard ball map $\beta: S_{\partial\Omega}\Omega \to S_{\partial\Omega}\Omega$ is such that $\beta(u) = v$ when v follows directly after u along the same (oriented) geodesic.

Assume that $\partial\Omega$ is C^1. Denote arc length along $\partial\Omega$ by s and the angle of the tangent $t(s)$ with the horizontal axis [the direction (1, 0)] by $\phi(s)$. This angle is defined (on the unit circle) once an orientation of $\partial\Omega$ is chosen and we choose the counterclockwise orientation here. The point $(x(s), y(s)) \in \partial\Omega$ is given by

$$x(s) = \int_{s_0}^{s} \cos(\phi(t))\, dt, \qquad y(s) = \int_{s_0}^{s} \sin(\phi(t))\, dt$$

We call Ω strictly convex when the tangent angle is strictly increasing with s. Then the tangent angle determines the point s on $\partial\Omega$ uniquely.

Assume that the initial trajectory of the billiard ball beginning at s forms the angle θ with the forward (counterclockwise) tangent to $\partial\Omega$ at s. The angle θ is called the angle of incidence at s.

To find the image (s_1, θ_1) of (s, θ) under the billiard ball map we must solve

$$\int_{s}^{s_1} \cos(\phi(t))\, dt = l \cos(\phi + \theta), \qquad \int_{s}^{s_1} \sin(\phi(t))\, dt = l \sin(\phi + \theta) \quad (1.1)$$

for s_1 (and l). Then with $\phi_1 = \phi(s_1)$ and $\phi = \phi(s)$, $\theta_1 = \phi_1 - \phi - \theta$.

2. PERIODIC POINTS

The orbit corresponding to a closed trajectory of the billiard ball passing through (s, θ) is a periodic orbit for β, and when n is the least positive integer with $\beta^n(s, \theta) = (s, \theta)$, then n is called the (primitive) period of the orbit.

To any orbit we associate a rotation number (if it exists) and to a periodic orbit we also associate a winding number.

Definition 1. Assume that the tangent angle ϕ is defined along the region's boundary $\partial\Omega$. The orbit starting at (s_1, θ_1) (with $\theta_1 \geq 0$ by convention) has *rotation number* ω when

$$\omega = \frac{1}{2\pi} \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n+1} (\phi_{j+1} - \phi_j)$$

where $(s_{j+1}, \theta_{j+1}) = \beta(s_j, \theta_j)$, $\phi_j = \phi(s_j)$, and the tangent angle is taken so that the differences are always nonnegative (and at most 2π). The rotation number is defined only when the above limit exists.

For example, the rotation number is between 0 and 1, the rotation number for $\theta = 0$ is zero, and the rotation number for $\theta = \pi$ is one.

Definition 2. A periodic orbit with period n has *winding number m* when

$$\sum_{j=1}^{n+1} (\phi_{j+1} - \phi_j) = 2\pi m$$

We denote the collection of all geodesics with period n and winding number m by $\Gamma(m, n)$ (the period is not assumed primitive).

We recall Poincaré's argument[10] that there is always at least one periodic orbit with each period and allowable winding number. Such a periodic orbit $\{(s_1, \theta_1),..., (s_n, \theta_n)\}$ can be found by examining the ordered collection of points $\{s_1,..., s_n\}$. For each fixed $1 \leqslant k \leqslant n-2$, set $x = s_k$, $y = s_{k+1}$, $z = s_{k+2}$. If $k = n-1$, then take $z = s_0$, and if $k = n$, then set $y = s_1$ and $z = s_2$. Let $L = L(y; x, z)$ be the sum of the lengths of the line segment between x and y and the line segment between y and z. By direct calculation, the collection $\{s_1,..., s_n\}$ defines a periodic orbit for the billiard ball map exactly when $\partial L/\partial y = 0$ holds for each k. The angles in the orbit are those θ_k so that the line through s_k along θ_k passes through s_{k+1}.

Theorem.[10] Assume that Ω is a strictly convex domain in $\mathbf{R}^2$ whose boundary is C^1. Fix integers $n \geqslant 2$ and $m \leqslant n/2$. Then $\Gamma(m, n) \neq \varnothing$.

Proof. Consider all oriented collections of n points, $\{s_1,..., s_n\}$, with $s_k \in \partial\Omega$, which wind around $\partial\Omega$ m times when we take $s_{n+1} = s_1$. Because $m \leqslant n/2$, this collection is not empty. Let M be the sum of the lengths of the line segments $\overline{s_k, s_{k+1}}$ (where when $k = n$ we take $s_{n+i} = s_i$). Then M is differentiable in each of the s_k and bounded above, and thus there is a collection $\{q_1,..., q_n\}$ that maximizes M. For each fixed $1 \leqslant k \leqslant n-1$, $L = L(s_{k+1}; s_k, s_{k+2})$ has a critical point (as a function of s_{k+1}), because M is maximized. Thus $\{q_1,..., q_n\}$ is a periodic orbit for the billiard ball map. The collection $\{q_1,..., q_n\}$ still has period n (the length M would be diminished if two q_k's coincided), and still has winding number m. This completes the proof of the theorem.

Remark 3. When $\partial\Omega$ is C^1 and strictly convex, the length function L is C^1 as a function of each of the points $\{s_1,..., s_n\}$. Thus it follows from the proof of the theorem above that if $\gamma_0 \in \Gamma(m, n)$ is an orbit in Ω and K_t, $0 \leqslant t < b$, are strictly convex C^1 curves with $K_t \to \partial\Omega$ as $t \to 0^+$, then for each t, K_t has a $\Gamma(m, n)$ orbit γ_t so that $\gamma_t \to \gamma_0$ and $L(\gamma_t) \to L(\gamma_0)$ as $t \to 0^+$. Here $K_t \to \partial\Omega$ as C^1 curves in the plane parametrized by the tangent angle, and $\gamma_t \to \gamma_0$ as curves in the plane.

3. LENGTH SPECTRUM

The length of geodesic segments used in the proof above also provides a generating function for the canonical relation associated with the fundamental solution for the Dirichlet problem for the Laplacian Δ in Ω. (For the development see refs. 8 and 3–6 and for a summary in dynamical terms see ref. 11.) Consider the eigenvalues λ^2 of

$$\begin{cases} \Delta u = \lambda^2 u & \text{in} \quad \Omega \\ u|_{\partial\Omega} = 0 \end{cases}$$

and define the enumerating measure

$$\sigma(\nu) = \sum \delta(\nu - \lambda)$$

Then it follows from Anderson and Melrose[2] that

$$\text{singular support}(\hat{\sigma}) \subset -\mathscr{L} \cup \{0\} \cup \mathscr{L}$$

where $\mathscr{L}$ is the length spectrum consisting of the lengths of closed geodesics and any multiple of the length of the boundary.

Marvizi and Melrose give a partial converse to this result. Let $t_{m,n}$ and $T_{m,n}$ be (respectively) the infimum and supremum of the lengths of geodesics in $\Gamma(m, n)$.

Theorem.[8] Assume that Ω is a smoothly (C^∞) bounded, strictly convex planar domain, and that there exists an N such that for $n > N$

$$L(\gamma_{m,p}) \neq t_{1,n}, \qquad \forall \gamma_{m,p} \in \Gamma(m, p), \quad m > 1$$

Then $t_{1,n} \in \text{sing.supp.}(\hat{\sigma})$. The same statement holds with $T_{1,n}$ replacing $t_{1,n}$ and with Neumann or Robin boundary conditions.

The restriction placed on lengths in the theorem above is called "noncoincidence." This condition is used as follows. The singular support of $\hat{\sigma}$ is decomposed, using oscillatory integrals, into parts $\hat{\sigma}_j$ for which it turns out that the phase is stationary at the lengths of closed geodesics with period j. This and a lemma due to H. Soga show that $t_{1,n} \in \text{sing.supp.}(\hat{\sigma}_n)$ (this is the singular spectrum[8]). To conclude here that $t_{1,n} \in \text{sing.supp.}(\hat{\sigma})$, the noncoincidence condition is needed (to avoid cancellation).

The use of the theorem above is that the singular spectrum determines geometric quantities associated to the region Ω. As will be seen in Section 7, one can extract the shape of certain domains from invariants which are determined by asymptotics for the lengths $t_{1,n}$. If noncoincidence holds, then these lengths are included in the singular spectrum. However, if other lengths, of $\gamma_{m,n}$ with $m>1$, approach $L(\partial\Omega)$, then it might be impossible to read $t_{1,n}$ from the length spectrum. In fact, the asymptotic behavior of $t_{1,n}$ might be impossible to read from the singular spectrum because other lengths might also appear. This problem is avoided by ref. 8 via the requirement that $\exists\delta>0$ with

$$(L(\partial\Omega)-\delta, L(\partial\Omega))\cap\mathscr{L}(\Omega)\subset\bigcup_{n=2}^{\infty} L(\Gamma(1,n))$$

We will call noncoincidence with this additional requirement *strong noncoincidence.*

4. NON-COINCIDENCE

For use with the theorems of Section 3 and 8, one wishes to extend the class of regions for which strong noncoincidence holds. In ref. 8 it is noted that for C^∞ boundaries coincidence can occur only for small winding numbers, and that strong noncoincidence holds for a dense set in the C^∞ topology including a neighborhood of circles. This neighborhood, however, is not known to include a neighborhood of any ellipses. The purposes of this section are to confirm the observations of ref. 8, to extend the class of regions with the strong noncoincidence property, and to further identify those orbits whose lengths may lead to coincidence or approach $L(\partial\Omega)$.

Proposition 4.[8] Assume that $\partial\Omega$ is C^1 and strictly convex. Fix $m\geqslant 1$ and for $n\geqslant 2m$ let $\gamma_{m,n}$ be any closed geodesic with winding number m and period n. Then as $n\to\infty$, $L(\gamma_{m,n})\to mL(\partial\Omega)$.

Proof. Assume for the moment that $\partial\Omega$ is C^3, with radius of curvature $\chi(s)$. Equation (1.1) gives $\phi_1\sim\phi+O(\theta)$, so $\phi_1=\phi+b\theta+O(\theta^2)$, with b a constant. Replacing $\phi(t)$ in the integral in Eq. (1.1) by its Taylor series and eliminating l, we get

$$\begin{aligned}\tan(\phi+\theta) &= \frac{1}{b\chi\cos\phi\cdot\theta}\,b\chi\sin\phi\cdot\theta\\ &\quad+\frac{1}{2}b^2\left(\dot\chi\sin\phi+\chi\cos\phi-\dot\chi\sin\phi+\frac{\chi\sin^2\phi}{\cos\phi}\right)\theta^2+O(\theta^3)\\ &= \tan\phi+\frac{1}{\cos^2\phi}\,\theta+O(\theta^2)\end{aligned}$$

This yields $b=2$ and consequently, for the rotation number ω (which is well defined since $\gamma_{m,n}$ is periodic), $\omega \sim \theta/\pi + O(\theta^2)$, and for the length of the line segment $l(s) \sim s_1 - s + O(\theta^2)$. It follows that

$$0 < (s_1 - s) - l(s) \leqslant c(s)\, \pi^2 \omega^2 \tag{4.1}$$

where the numbers $c(s)$ are uniformly bounded for $s \in \partial\Omega$, say by C. Hence

$$0 < mL(\partial\Omega) - L(\gamma_{m,n}) \leqslant nC\omega^2 = \frac{Cm^2}{n} \tag{4.2}$$

Whenever $\partial\Omega$ is C^1 and strictly convex, it can be approximated by C^3 curves K_t, $0 \leqslant t < b$, with radii of curvature $\chi_t > 0$. As noted in Remark 3, the lengths, winding numbers, and periods of closed geodesics are also approximated. Since (4.2) is independent of χ_t the proof of the proposition is complete. ∎

The following proposition is not needed for Theorem 6, but is included as an estimate on those orbits whose lengths might (not) approach $L(\partial\Omega)$.

Proposition 5. There is a number $t_0 > 0$, depending only on Ω, so that for each m and n with $m \leqslant n/2$

$$mt_0 \leqslant L(\gamma_{m,n}) \leqslant \frac{n}{2}\, T_{1,2} \tag{4.3}$$

Proof. Since $\gamma_{m,n}$ has n segments and each segment is of length at most $\frac{1}{2}T_{1,2}$, it is clear that $L(\gamma_{m,n}) \leqslant (n/2)\, T_{1,2}$. Consider the largest (ordinary) circle inscribed in $\partial\Omega$, and let t_0 be twice its diameter. Then the lower estimate holds for geodesics in (the region bounded by) the circle. In fact, for any oriented collections of n points $\{\bar{\phi}_1, ..., \bar{\phi}_n\}$ which wind around the (inscribed) circle m times, the sum of the lengths of the segments connecting the points is at least mt_0. Consider then a trajectory with points $\{s_1, ..., s_n\}$, with $s_k \in \partial\Omega$, which winds around $\partial\Omega$ m times, and take $\phi_k = \phi(s_k)$ to be the tangent angles. Then taking the points in the inscribed circle to be $\bar{\phi}_k = \phi_r$, $\{\bar{\phi}_1, ..., \bar{\phi}_n\}$ winds around the circle m times and for the lengths, taken in the plane, $\sum |\bar{\phi}_{k+1} - \bar{\phi}_k| \leqslant \sum |s_{k+1} - s_k|$. To confirm the last inequality consider a starting point ϕ_k and for $j > k$ examine the line segment from ϕ_k to ϕ_j. If this line segment intersects the circle, then $\sum |\bar{\phi}_{k+l+1} - \bar{\phi}_{k+l}| \leqslant \sum |s_{k+l+1} - s_{k+l}|$, the sums being over $0 \leqslant l \leqslant k-j$. If the line segment does not intersect the circle and $\phi_j - \phi_k > 2\pi$, then the orbit encloses the circle and, again, $\sum |\bar{\phi}_{k+l+1} - \bar{\phi}_{k+l}| \leqslant \sum |s_{k+l+1} - s_{k+l}|$. Hence $mt_0 \leqslant L(\gamma_{m,n})$. ∎

Theorem 6. Assume that a planar region Ω has C^1 boundary and is strictly convex in the sense that the tangent angle determines a point on the boundary. Then the region satisfies noncoincidence.

Proof. By Proposition 4, or alternatively by Proposition 5, $\exists M$ so that for $m \geqslant M$, $L(\gamma_{mm}) > L(\partial\Omega) + 1$. We thus assume from now on that $m < M$.

By Proposition 4 there is an N with $L(\gamma_{m,n}) > 1.5 L(\partial\Omega)$ for any choice of $2 \leqslant m < M$ whenever $n > N$. ▌

Remark. The proof of Theorem 6 and Eq. (4.2) can be used to show that strong noncoincidence holds in special cases. An example of this is given in Corollary 7. However, the example of an ellipse for which the length of the billiard orbit of period two along the minor axis equals half the circumference shows that strong noncoincidence does not hold for convex regions in general. [In this example the length of the (4,2) orbit along the minor axis coincides with the length of the boundary, and one can perturb this ellipse so that a (4,2) orbit with primitive period 4 has length coinciding with that of the boundary.]

5. EXAMPLE OF A SMOOTH NONCONVEX DOMAIN WITH COINCIDENCE

Consider the planar region built on a rectangular grid as follows. The base is of length 3, the left side and top are of length 2, and on the right side a collection of "stairs" each half the size of the previous one is constructed (see Fig. 1). The tallest stair has height 1, the next has height 0.5, etc.

The total length of the boundary is 10, and for each integer $k \geqslant 1$ there is a $(5 \cdot 2^k,\ 10 \cdot 2^k)$ orbit with segments of length 2^{-k} inside the stair with

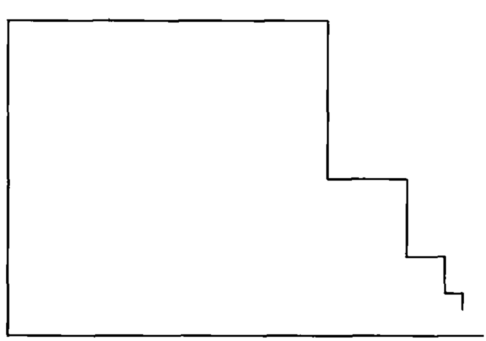

Fig. 1. Base shape for a smooth example of coincidence.

height 2^{-k}. This orbit consists of repetitions of the period-2 orbit with segments whose endpoints lie on the base and top of the stair and its length is 10.

It is clear that the region's boundary can be made smooth, by changing the boundary where the horizontal and vertical segments meet, but not changing the horizontal segments at the top of the stairs and thus retaining the orbits of length 10 above. Thus strong noncoincidence fails.

Alternatively, one can replace the straight-line segments forming the top of the stairs with concave-down elliptic segments, say, so that the lengths of the $(5 \cdot 2^k, 10 \cdot 2^k)$ orbits still coincide with the length of the boundary, but the orbits have primitive period $10 \cdot 2^k$ (even though this is not required for coincidence). Note that the lengths of the trajectories and of the boundary are increased slightly here, but that the adjustment for coincidence can be made since one can always increase or decrease the length of the base and top near the left side. Similarly, one can arrange (say, by increasing the curvature of the elliptic segments) to have the lengths of the $(5 \cdot 2^k, 10 \cdot 2^k)$ orbits not to coincide with the length of the boundary, but to approach the length of the boundary as $k \to \infty$.

With any fixed length in mind, periodic orbits in the "stairs" with winding numbers larger than any lower bound prescribed in advance can be chosen to approach that length. It follows that any noncoincidence condition fails. It should be noted that even in the smoothed version of this example the lengths of those $(1, n)$ orbits that exist will not approach the length of the boundary (which is not surprising since this region is not convex).

6. EXAMPLE OF A CONVEX DOMAIN WITH COINCIDENCE

This region consists of a circular arc of radius R with endpoints B and C and two line segments meeting at a point A (see Fig. 2). The segment $\overline{AB}$ meets the circular arc at a right angle. The desired billiard trajectory begins

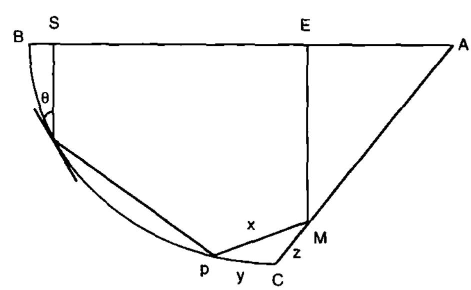

Fig. 2. Base shape for a convex example of coincidence.

at a right angle to the segment $\overline{AB}$ at the point S near B, is reflected by the arc segment (BC), and is reflected by the line segment $\overline{CA}$ at a point M so as to meet $\overline{AB}$ at a right angle again, at the point E. In view of the two right angles, the trajectory is closed.

Let α denote the angle between the billiard ball's trajectory and the line segment $\overline{CA}$, and let θ denote the trajectory's angle with the circular arc. To obtain the trajectory above with n reflections off of the circular arc we need to choose the segment $\overline{CA}$ so that $\alpha = -n\theta + \pi/2$, which is possible (consider fixing θ and n and the point C so that $0 < n\theta < \pi/2$ and varying α between the two extremes $\alpha = 0$ and $\alpha = \pi$).

Let x denote the length of the segment of the billiard trajectory with one endpoint at the point p on the circular arc and one at M. Now one wishes to change the portion of the region's boundary outside this last segment so that similar closed orbits with smaller angles of incidence θ will be created without destroying the existing orbit. One also wishes to arrange the lengths of these orbits to approach the length of the region's boundary.

To see that these can be achieved, we calculate the lengths of the curves which are involved. Let y denote the length of the arc segment between p and C and z denote the length of the line segment $\overline{CM}$. The length of the region's boundary (without the modification near x) is

$$L(\partial\Omega) = |\overline{ME}| \left(\frac{1+\sin\alpha}{\cos\alpha}\right) + |\overline{SE}| + R(1-\cos\theta) + R\theta + 2(n-1)\,R\theta + y + z$$

The length of the trajectory is

$$L(\gamma) = 2[\,|\overline{ME}| + x + R\sin\theta + 2(n-1)\,R\sin\theta]$$

The following relations also hold:

$$|\overline{ME}| = R\sin(\theta)[1 + 2\cos(2\theta) + \cdots + 2\sin(2(n-1)\,\theta)] + x\cos(2n\theta)$$

$$|\overline{SE}| = 2R\sin(\theta)[\sin(2\theta) + \cdots + 2\sin(2(n-1)\,\theta)] + x\sin(2n\theta)$$

The question is, then, whether x and θ can be adjusted to keep $L(\partial\Omega)$ constant and have $n \to \infty$. The answer is yes, and to see this we must analyze how y changes when θ changes. For a circular arc with incident angle θ, the length of a segment of the geodesic of period n is

$$l = 2R\sin(\theta) = 2R\sin\left(\frac{\pi}{n}\right) \sim \frac{2R\pi}{n} - \frac{R\pi^3}{3}n^{-3} + O(n^{-5})$$

and the difference between the arc length along the boundary and the length of the geodesic segments for n reflections is

$$L(\partial\Omega) - L(\gamma_n) = n\frac{2\pi R}{n} - 2R\sin\left(\frac{\pi}{n}\right) \sim \frac{\pi^3 R}{3} n^{-2} + O(n^{-4})$$

Hence $L(\partial\Omega) - L(\gamma_n) \ll l$ for large n and the above adjustment can be made. [One starts with $L(\partial\Omega)$ a bit larger than the length of the orbit starting at S, and the adjustment to $\partial\Omega$ is made by moving C toward p along the circular arc and replacing the line segment $\overline{MC}$ with a number of line segments. Corresponding to a smaller angle θ a larger angle α is needed, so the line segments replacing $\overline{MC}$ keep $\partial\Omega$ convex. The boundary is not adjusted on its portion $MABp$].

This construction gives billiard trajectories so that $L(\gamma_{2,n}) \to L(\partial\Omega)$ and thus it may be impossible to calculate $L(\partial\Omega)$ from the singular spectrum.

Note that here we have used the fact that the reflected trajectories meet the segment $\overline{BA}$ at right angles (specifically near B), and so in this example the boundary cannot be made C^1.

7. LENGTHS AND SPECTRAL INVARIANTS

The results in this and the next section follow refs. 8 and 1. Assume henceforth that $\partial\Omega$ is C^2 and has positive curvature.

A planar curve C whose tangents are invariant under reflection at the boundary defines an invariant circle for the billiard ball map and is also called a caustic. It is clear that a line segment tangent to C is part of a geodesic whose every segment is tangent to C and thus that near $\partial\Omega$ there

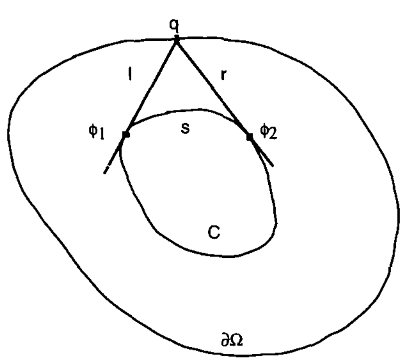

Fig. 3. An invariant circle.

is a one-to-one correspondence of C^1 convex caustics and invariant sections for the boundary map. Such curves C are characterized by the Lazutkin parameter.

For $C \subset \Omega$ a strictly convex curve and a point $q \in \partial\Omega$, there are precisely two points $\phi_1, \phi_2 \in C$ so that the tangent lines to C at ϕ_1 and ϕ_2 go through q. We denote the lengths of the line segments between ϕ_1 and q and between q and ϕ_2 by r and l, and the arc length along C between ϕ_1 and ϕ_2 by s, and define the *Lazutkin parameter* V of C and $\partial\Omega$ at q by $V = r + l - s$. (See Fig. 3.)

Lemma.[7] Assume that $\partial\Omega$ is C^1 and strictly convex. A strictly convex C^1 closed planar curve homotopic to $\partial\Omega$, $C \subset \Omega$, is a caustic iff the Lazutkin parameter of C and $\partial\Omega$ at $q \in \partial\Omega$ is independent of the point q. In fact,

$$\frac{dV}{du}(u) = \cos\theta_+(u) - \cos\theta_-(u)$$

where u denotes arc length along $\partial\Omega$ and $\theta_+(u)$ and $\theta_-(u)$ are the angles formed at u with the tangent to $\partial\Omega$ by the outgoing and incoming geodesic segments.

Let ϕ be a point on $\partial\Omega$ (q above in terms of tangent angle) and $\phi_1 < \phi < \phi_2$ be the two points on C where tangent lines to C pass through ϕ, and let k denote the curvature function of $\partial\Omega$.

The parameter $\Delta := \phi_2 - \phi = \phi - \phi_1$ is essentially the rotation number ω. More precisely,

$$\frac{1}{2\pi}\int_0^{2\pi} 2\,\Delta(\phi)\,d\phi = \omega$$

and

$$\omega^2 \sim (12)^{2/3}\,G(k)\,V^{2/3} + O(V^{4/3}), \qquad G(k) = \int_0^{2\pi} k^{2/3}(\phi)\,d\phi$$

Our variant of Birkhoff's invariant is $G(k)$.

After inversion and integration, we obtain a formal power series at $V = 0$ for the length of an invariant curve $L(C)$ in terms of $V^{2/3}$. The coefficients of this series and G above are the caustics' invariants. That is, with k and f denoting the curvatures of $\partial\Omega$ and of C,

$$k(\phi) \sim \sum_{j=0}^{\infty} E_j(f)(\phi)\,V^{2j/3}$$

and $E_0(f) = f$, so there are M_j with

$$f(\phi)^{-1} \sim \sum M_j(k)(\phi)\, V^{2j/3}$$

and for the length of C

$$L(C) = \int_0^{2\pi} f(\phi)^{-1} d\phi \sim \sum_{j=0}^{\infty} J_j V^{2j/3} \tag{7.1}$$

where

$$J_j = J_j(k) = \int_0^{2\pi} M_j(k)(\phi)\, d\phi \tag{7.2}$$

are the caustics' invariants. These invariants can be calculated from (7.1) and the geometry.

Beginning with the E_j, one obtains

$$\begin{aligned} k(\phi) \sim f(\phi) &+ (\tfrac{3}{2})^{2/3} (\tfrac{1}{6} f^{2/3} f'' - \tfrac{2}{9} f^{-1/3}(f')^2 - \tfrac{1}{2} f^{5/3})\, V^{2/3} \\ &+ (\tfrac{3}{2})^{1/3} [\tfrac{1}{80} f^{4/3} f^{(4)} - \tfrac{1}{15} f^{1/3} f' f^{(3)} - \tfrac{1}{60} f^{1/3}(f'')^2 + \tfrac{1}{9} f^{-2/3}(f')^2 \\ &- \tfrac{11}{40} f^{4/3} f'' - \tfrac{1}{27} f^{-5/3}(f')^4 + {}^1 f^{1/3}(f')^2 + \tfrac{21}{80} f^{7/3}]\, V^{4/3} + O(V^{6/3}) \end{aligned}$$

Further calculations give

$$J_0 = \int_0^{2\pi} k^{-1}(\phi)\, d\phi = L(\partial\Omega), \qquad J_1 = -(\tfrac{3}{2})^{2/3} \tfrac{1}{2} \int_0^{2\pi} k^{-1/3}(\phi)\, d\phi$$

and

$$J_2 = \tfrac{1}{720} (\tfrac{3}{2})^{1/3} \int_0^{2\pi} (8k^{-5/3}(k')^2 + 9k^{1/3})(\phi)\, d\phi$$

8. RELATIONS ARISING FROM THE DYNAMICS ON A CAUSTIC

Consider a geodesic γ (which we will view as a biinfinite sequence in $S_{\partial\Omega}\Omega$) and a distinguished starting point ξ in γ. Let $\angle(\beta^n(\xi), \beta^{n-1}(\xi))$ denote the angle (in $T\mathbf{R}^2$) between the successive segments of the geodesic, and let $|\,,\,|$ denote distance in $\mathbf{R}^2$. Set

$$\omega_m(\gamma, \xi) = \frac{1}{2\pi m} \sum_{n=1}^{m} \angle(\beta^n(\xi,), \beta^{n-1}(\xi))$$

and

$$A_m(\gamma, \xi) = \frac{1}{m} \sum_{n=1}^{m} |\pi \circ \beta^n(\xi), \pi \circ \beta^{n-1}(\xi)|$$

where π is the projection (to $\partial\Omega \subset \mathbf{R}^2$).

On a caustic C, A_m and ω_m are related to the Lazutkin parameter and the length of the caustic:

Proposition.[1] For an invariant circle C, γ a closed geodesic of period m tangent to C, and $\xi \in \gamma$,

$$mV(C, \partial\Omega) = mA_m(\gamma, \xi) - m\omega_m(\gamma, \xi) \cdot L(C)$$

where $L(C)$ is the length of C and V is the Lazutkin parameter.

Proposition.[1] For a geodesic γ tangent to a fixed invariant circle C,

$$\omega(\gamma) := \lim_{m \to \infty} \omega_m(\gamma, \xi)$$

is well defined. In fact, ω is independent of the choice of γ tangent to C.

Corollary.[1] For a geodesic γ tangent to a caustic C,

$$A(\gamma) := \lim_{m \to \infty} A_m(\gamma, \xi)$$

is well defined and independent of γ. Moreover,

$$V(C, \partial\Omega) = A(\gamma) - \omega(\gamma)\, L(C) \tag{8.1}$$

The boundary map $\delta\colon B^*\partial\Omega \to B^*\partial\Omega$ is defined as follows. When $u \in S_p\partial\Omega$ defines the incidence angle θ and $\xi \in B^*\partial\Omega$ with $\xi = \cos(\theta)$ and $v = \beta(u)$ defines the incidence angle θ', then $\xi' = \delta(\xi) \in B^*_{\pi(v)}\partial\Omega$ with $\xi' = \cos(\theta')$.

This map preserves the canonical, exact, two-form ν on $B^*\partial\Omega$, and Marvizi and Melrose calculated explicitly its pullback of a one-form α with $d\alpha = \nu$. This leads to spectral invariance of the wave invariants described below.

There is a function

$$\zeta \in C^\infty(B^*\,\partial\Omega) \qquad \text{with} \quad \delta = \exp(-\zeta^{1/2} H_\zeta) + \psi$$

where H_ζ is the Hamiltonian vector field of ζ, and ψ is a smooth symplectic map which fixes the boundary component $S^*_+\,\partial\Omega$ of $B^*\,\partial\Omega$ to all orders.

This determines the Taylor series for ζ at $S^*_+\,\partial\Omega$. (The Hamiltonian above is called the interpolating Hamiltonian.[9]) One can choose ζ so that $d\zeta \neq 0$ on all of $B^*\,\partial\Omega$ and so that $\zeta = 2$ on the boundary component $S^*_-\,\partial\Omega$.

With s denoting arc length along $\partial\Omega$, the one-form dz is defined on the curve $\{\zeta = c\}$ via $dz(H_\zeta) = 1$ and $z(s=0) = 0$. The function

$$I(c) = \int_{\{\zeta = c\}} dz$$

is C^∞ in c near 0, with a Taylor series at $c = 0$ which is independent of the choice of ζ. The wave invariants are the coefficients in this Taylor series, namely

$$I_{k+1} = \frac{d^k I}{dc^k}(0), \qquad k = 0, 1, 2, \ldots$$

Subsequently, ref. 8 shows that, with ν denoting the symplectic form of $B^*\,\partial\Omega$,

$$\delta^* z \equiv z - \zeta^{1/2}, \qquad \delta^*\zeta \equiv \zeta, \qquad \nu = d\zeta \wedge dz$$

where equivalence is in the sense of Taylor series at $\zeta = 0$.

Marvizi and Melrose now set

$$F(\zeta) = \int_0^\zeta I(u)\,du, \qquad \alpha = F(\zeta)\,d(z/I(\zeta)) + ds$$

Then α is a 1-form on $B^*\partial\Omega$ with $d\alpha = \nu$.

Further calculations show that

$$\delta^*\alpha - \alpha = d\left(-\frac{F(\zeta)}{I(\zeta)}\zeta^{1/2} + \frac{2}{3}\zeta^{3/2} + \delta^* s - s + h\right)$$

where h vanishes to all orders at $S^*_+\,\partial\Omega$. This determines the generating function for the length in Poincaré's theorem from Section 2 explicitly.

On a simple closed geodesic with period n, $\gamma_{1,n}$,

$$I(\zeta) \equiv \zeta^{1/2}$$

and thus the length of $\gamma_{1,n}$ is

$$L(\gamma_{1,n}) = L(\partial\Omega) - F(\zeta) + \tfrac{2}{3}\zeta I(\zeta)$$

Hence ζ and $I(\zeta)$, computed for closed simple geodesics, have asymptotic expansions in terms of the rotation number squared, that is, $1/n^2$ for $\gamma_{1,n}$.

Differentiating $I(\zeta)$ and $L(\gamma_{1,n})$ (implicitly), one concludes that the power series for ζ and $I(\zeta)$ as functions of the rotation number (squared) are determined by the length spectrum.

Under the strong noncoincidence assumption (see pp. 490 and 495 of and the discussion in Section 3 above), the length spectrum is determined by the spectrum for the Laplacian with Dirichlet, Neumann, or Robin boundary conditions;[6,8] the assymptotics above are spectral invariants. Using the relations involving invariant curves above and the spectral invariance of the wave invariants, we obtain that the same invariance holds for the caustics' invariants.

Theorem.[1] The caustics' invariants are determined by the spectrum of the Laplacian in a C^∞ planar domain that satisfies the strong noncoincidence condition.

Theorem.[1] Assume that a planar curve is C^∞, closed, has strictly positive curvature, has fixed first wave or caustics' invariant (the length L) and second wave or caustics' invariant (J_1), and maximizes $-6G$. Then the curve is an ellipse.

Corollary 7. An ellipse for which the length of the minor axis exceeds one-fourth of the length of the ellipse is determined by the spectrum of the Laplacian among C^∞ planar domains whose boundaries have strictly positive curvature and are near the ellipse.

Proof. Since the shortest of the orbits with winding number at least two lies along the minor axis of the ellipse, it follows that $M=2$ and $N=4$ can be used in Eq. (4.2) and the proof of Theorem 6. Thus strong noncoincidence holds for the ellipse. By Remark 3, for planar domains whose boundaries have strictly positive curvature near that of the ellipse, $M=2$ and $N=4$ can also be used in (4.2). Now the theorem above establishes the corollary.

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